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ON CENTRAL LIMIT THEOREMS IN GEOMETRICAL PROBABILITY

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We prove central limit theorems and establish rates of convergence for the following problems in geometrical probability when points are generated in the $[0,1]^2$ cube according to a Poisson point process with parameter n:

1. The length of the nearest graph $N_{k,n}$, in which each point is connected to its kth nearest neighbor.
2. The length of the Delaunay triangulation Del_n of the points.
3. The length of the Voronoi diagram Vor_n of the points.

Using the technique of dependency graphs of Baldi and Rinott, we show that the dependence range in all these problems converges quickly to 0 with high probability. Our approach also establishes rates of convergence for the number of points in the convex hull and the area outside the convex hull for points generated according to a Poisson point process in a circle.

1. Introduction. In a pioneering paper by Beardwood, Halton and Hammersley [5] and continued in Steele [15], it was shown that the lengths L_n of several combinatorial optimization problems (the traveling salesman, minimum matching, minimum spanning tree, minimum Steiner tree, etc.) satisfy laws of large numbers when their input consists of n random i.i.d. points X_k, $k = 1, \dots, n$, in the cube $[0,1]^2$.

It is also believed (see, e.g., Steele [16]), but yet unknown, that they satisfy central limit theorems (CLTs), because although the edges of the foregoing optimal graphs are not independent, the dependence "seems to be local." In trying to make this intuitive idea precise, we succeeded in proving CLTs for the length of three graphs, which are among the most fundamental constructions in computational geometry (see, e.g., Preparata and Shamos [13]). In the following definitions the length of a graph is defined to be the sum of the lengths of its edges. The three graphs are:

1. The length $N_{k,n}$ of the kth nearest graph, in which each point is connected to its kth nearest neighbor.

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2. The length Vor_n of the Voronoi diagram of the points, which is the collection of Voronoi cells for each point. Given a collection of points, a Voronoi cell around a point O is the set of all points that are closer to O than to any other point (see Figure 1, dotted lines).
3. The length Del_n of the Delaunay triangulation of the points, which is the graph defined on the given points with an edge between them if they are Voronoi neighbors (see Figure 1, solid lines). Assuming that no three points are collinear and no four points lie in the same circle, which happens with probability 1 if the points are generated according to a Poisson point process, this graph is indeed a triangulation; that is, a subdivision of the convex hull of points in triangles.

Both the Voronoi diagram and the Delaunay triangulation are fundamental constructions in computational geometry because many algorithms for solving geometrical problems are based on them. In particular, the Delaunay triangulation is quite important for the minimum spanning tree (MST) because it contains the MST as a subgraph; that is, points that are neighbors in the MST must also be Voronoi neighbors (see Figure 2). This property leads to an efficient algorithm for the MST in the plane because one first constructs the Delaunay triangulation in $O(n \log n)$ time and then runs the greedy algorithm on its graph, leading to an $O(n \log n)$ algorithm for the MST as well (see, e.g., Preparata and Shamos [13]).

We believe that our results give some partial insight on why a CLT might hold for the MST as well. Ramey [14] has attempted to prove a CLT for the MST, but his approach, although very interesting, did not succeed because he needed some unproven, but plausible, lemmas from continuous percolation.

The CLT for $N_{1,n}$ has already been obtained by Bickel and Breiman [6], who used complicated fourth moment estimates. They wrote: "Our proof is

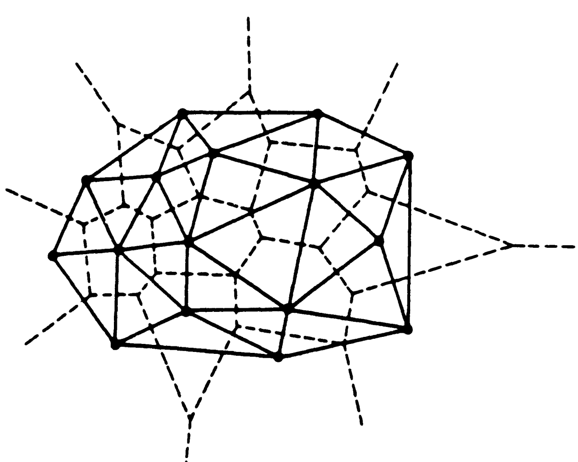

FIG. 1. *The Voronoi diagram and the Delaunay triangulation.*

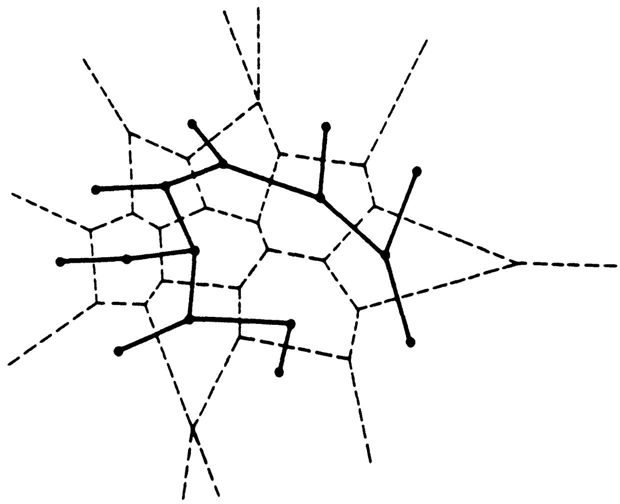

FIG. 2. *The MST is a subgraph of the Delaunay triangulation.*

long. We believe that this is due to the complexity of the problem. Nearest neighbor distances are not independent." We used instead a simple conditioning argument that establishes that the problems exhibit finite dependence. In addition, the current approach not only leads to CLTs, but it establishes rates of convergence.

Our results are established for the Poisson model, which is defined as follows.

POISSON MODEL. Let X_i, $i = 1, \dots, N_n$, be the points of a Poisson process with intensity n on R^2 that lie in a two-dimensional cube $[0,1]^2$, so that N_n is a Poisson random variable with mean n.

This article is structured as follows. Section 2 contains the proof of our main result that for all three problems,

$$\lim_{n \to \infty} P\left\{ \frac{L_n - E[L_n]}{\sqrt{\mathrm{Var}[Ln]}} \leq x \right\} = \Phi(x),$$

where $\Phi(x)$ is the cumulative distribution of a standard $N(0,1)$ normal. We also establish that the rate of convergence is

$$\left| P\left\{ \frac{L_n - E[L_n]}{\sqrt{\mathrm{Var}[L_n]}} \leq x \right\} - \Phi(x) \right| = O\left(\frac{(\log n)^{1+3/4}}{n^{1/4}} \right).$$

Moreover, we remark that our approach obtains the rate of convergence for the CLTs on the number of points in the convex hull (obtained by Groeneboom [8]) and the area outside the convex hull (obtained by Hsing [9]) for points generated according to a Poisson process of rate n in a circle.

It is well known that the expected lengths $E[L_n]$ of the kth nearest neighbor graph, the Delaunay triangulation and the Voronoi diagram in the two-dimensional cube $[0, 1]^2$ satisfy

$$\lim_{n \to \infty} \frac{E[L_n]}{n^{1/2}} = \beta.$$

The constants β have been explicitly computed by Miles [11], which is in sharp contrast with the problems studied in references 5 and 15 with the exception of the MST constant that we calculated recently [1]. We review the relevant results in Section 3. In the final section we include some concluding remarks and discuss some open questions.

2. Central limit theorems. In this section we prove CLTs for the three problems we consider. We first outline the methodology and comment on its applicability to other combinatorial problems.

2.1. *Intuitive idea.* We identify a certain event A_n, such that $P\{A_n\} \to 1$, whose occurrence implies independence of the configuration at points that are distant enough. More precisely, there is a cutoff distance, such that if the event A_n happened, then deterministically, the configuration around a given point is not influenced by that of points further than the cutoff distance.

For all three problems, A_n is the event that, in the subdivision of $[0, 1]^2$ into $O(n/\log n)$ equal subcubes, each subcube contains at least one and at most $O(\log n)$ points of the Poisson process. If all the neighboring subcubes around a point are nonempty, then only a finite number of them determine the kth nearest neighbors or the Voronoi neighbors of a point. Thus, conditioned on the event A_n, the three problems exhibit "m-dependence" (at the subdivision level) for some finite m. Applying the theory of dependency graphs [3] yields the CLTs.

As mentioned in the introduction, our approach was inspired by the Ph.D. thesis of Ramey [14], who attempted a similar approach for the minimum spanning tree. In order to see how our approach might generalize to the MST, consider the event $B_{n,k}$ that there exist two points that are Voronoi neighbors and lie at some distance l of each other, but can also be connected by a chain of edges that are all shorter than l, and the shortest such chain between the two points use more than k edges for some large k. Two such points will not be neighbors in the MST and the decision not to connect them *is affected* by the position of far away points. Let $B'_{n,k}$ be the complement of the event $B_{n,k}$.

Conditioning in this case on $A_n \cap B'_{n,k}$ makes the problem m-dependent with m finite and thus shows that if $P\{B_{n,k}\} \to 0$ as $n \to \infty$, the CLT for the MST follows (modulo some technicalities). Further analogies with continuous percolation make the preceding hypothesis quite plausible, even though a rigorous proof has not been submitted.

To summarize, in the nearest neighbor, the Voronoi diagram and the Delaunay triangulation we will establish the event whose probability tends to

1 and that implies finite range dependence and thus the CLT. In contrast, for the MST we know the event, but we cannot show its probability tends to 1. Finally, for the traveling salesman and minimum matching problems we have not identified an event whose probability tends to 1 and which implies finite range dependence and thus the CLT.

2.2. *Dependency graphs.* The fundamental concept that captures the idea of local dependence is that of dependency graphs. Introduced in Petrovskaya and Leontovitch [12] and applied to several problems by Baldi and Rinott [3, 4] dependency graphs are defined as follows: Let X_a, $a \in V$, be a collection of random variables. The graph $G = (V, E)$ is said to be a *dependency graph* for X_a if for any pair of disjoint sets $A_1, A_2 \subseteq V$ such that no edge in E has one endpoint in A_1 and the other in A_2, the σ-fields $\sigma\{X_a, a \in A_1\}$ and $\sigma\{X_a, a \in A_2\}$ are mutually independent.

THEOREM 1 (Baldi and Rinott [4]). *Let $\{X_{an}, a \in V_n\}$ be random variables having a dependency graph $G_n = (V_n, E_n)$, $n \ge 1$. Let $S_n = \sum_{a \in V_n} X_{an}$, $\sigma_n^2 = \mathrm{Var}[S_n] < \infty$. Let D_n denote the maximum degree of G_n and suppose $|X_{an}| \le B_n$ for a constant B_n almost surely for all $a \in V_n$. Then*

$$\left| P\left\{ \frac{S_n - E[S_n]}{\sigma_n} \le x \right\} - \Phi(x) \right| \le 32(1 + \sqrt{6}) \left(\frac{|V_n| D_n^2 B_n^3}{\sigma_n^3} \right)^{1/2}, \tag{1}$$

where $|V_n|$ is the cardinality of V_n. Thus, if $(|V_n| D_n^2 B_n^3)/\sigma_n^3 \to 0$ as $n \to \infty$,

$$\frac{S_n - E[S_n]}{\sigma_n} \to \mathcal{N}(0,1).$$

2.3. *The CLT.* Let X_i, $i = 1, \dots, N_n$, be the points of a Poisson process with intensity n on R^2 that lie in a two-dimensional cube $[0,1]^2$, so that N_n is a Poisson random variable with mean n. Let L_n be the length of the graphs $N_{k,n}$, Del_n or Vor_n. We subdivide the cube $[0,1]^2$ into a set C of m^2 subcubes of equal size with $m = \lfloor (n/(c \log n))^{1/2} \rfloor$ with $c > 5/4$. Let $\lambda = n/m^2$. Because we selected $m = \lfloor (n/(c \log n))^{1/2} \rfloor$, we have that

$$\left(\frac{n}{c \log n} \right)^{1/2} - 1 < m \le \left(\frac{n}{c \log n} \right)^{1/2}.$$

The left inequality leads to $(n/(c \log n))^{1/2} < m + 1 \le 2m$; that is, $\lambda = n/m^2 < 2^2 c \log n$. The right inequality leads to $\lambda = n/m^2 \ge c \log n$.

The subcubes will be denoted by indices $\mathbf{i} = (i_1, \dots, i_d)$, where each i_j takes value from 1 to m. We decompose the total length L_n as

$$L_n = \sum_{\mathbf{i} \in C} L_{\mathbf{i}, n},$$

where $L_{\mathbf{i}, n}$ is the sum of all edges with both ends in the subcube $\mathbf{i}$ plus the sum of the portion of the edges within the subcube with only one endpoint in the subcube $\mathbf{i}$. Let $N_{\mathbf{i},n}$ be the number of points falling in the cube $\mathbf{i}$. Let A_n be the event that each subcube is nonempty and contains less than or equal to $\lceil e\lambda \rceil$ points; that is,

$$A_n = \bigcap_{\mathbf{i} \in C} \{1 \le N_{\mathbf{i},n} < \lceil e\lambda \rceil\}.$$

We first show the following lemma.

LEMMA 2.

$$\lim_{n \to \infty} P\{A_n\} = 1.$$

PROOF. It is easy to check that

$$P\{1 \le N_{\mathbf{i},n} < \lceil e\lambda \rceil\} \ge 1 - 2e^{-\lambda},$$

so by the independence of the $N_{\mathbf{i},n}$ we have

$$(1 - 2e^{-\lambda})^{m^2} \le P\{A_n\} \le 1. \tag{2}$$

Taking limits and using $\lambda \ge c \log n$, $c > 5/4$ we obtain the lemma. □

We introduce now a distance between subcubes $\mathbf{i}$ and $\mathbf{j}$:

$$d(\mathbf{i},\mathbf{j}) = \max_{1 \le r \le 2} \{|i_r - j_r|\}.$$

Let

$$S_{\mathbf{i},R} = \{\mathbf{j} \in C \colon d(\mathbf{i},\mathbf{j}) \le R\};$$

that is, $S_{\mathbf{i},R}$ denotes the sphere of subcubes of radius R around $\mathbf{i}$. Moreover, if A, B are two sets of subcubes, we let

$$S_{A,R} = \bigcup_{\mathbf{i} \in A} S_{\mathbf{i},R},$$

$$d(A,B) = \min_{\mathbf{i} \in A, \mathbf{j} \in B} d(\mathbf{i},\mathbf{j}).$$

The following proposition is the heart of our development of the CLT and captures the idea of "local dependence" in the three problems we consider.

PROPOSITION 3. *There exists a fixed number R such that, conditionally on the event A_n, for any pair of sets of subcubes A, B with $d(A,B) > R$, the σ-fields $\sigma\{L_{\mathbf{i},n}, \mathbf{i} \in A\}$ and $\sigma\{L_{\mathbf{i},n}, \mathbf{i} \in B\}$ are independent.*

PROOF. Let X_v be the location of point v and let $X_{\mathbf{i}}$ denote the set $\{X_v, X_v \in \mathbf{i}\}$; that is, the set of points in a subcube. We note first that the point process on $[0,1]^2$, obtained from the Poisson point process by conditioning on A_n, will retain the property that $X_{\mathbf{i}}$ will be independent.

For ease of exposition in the k nearest neighbor graph we present the case $k = 1$.

For the nearest neighbor graph ($k = 1$), we prove the proposition with $R = 4$. Note first that if point $X \in \mathbf{i}$ is the nearest neighbor of $Y \in \mathbf{j}$ or $Y \in \mathbf{j}$

is the nearest neighbor of $X \in \mathbf{i}$, then $d(\mathbf{i},\mathbf{j}) \leq 2$ because if $d(\mathbf{i},\mathbf{j}) \geq 3$, then a point in a subcube at distance 1 from $\mathbf{i}$ is guaranteed to be closer than Y (see also Figure 3).

Moreover, the nearest neighbor of $X \in \mathbf{i}$ belongs in one of the subcubes in $S_{\mathbf{i},2}$ and its identity is in no way affected by any point outside $S_{\mathbf{i},2}$. Note that the analog of this property fails to hold for the minimum spanning tree, matching and traveling salesman problems. However, conditionally on the event A_n, $\{L_{\mathbf{i},n}, \mathbf{i} \in A\}$ is some function $f\{X_{\mathbf{i}}, \mathbf{i} \in S_{A,2}\}$ and similarly $\{L_{\mathbf{i},n}, \mathbf{i} \in B\}$ is some function $f\{X_{\mathbf{i}}, \mathbf{i} \in S_{B,2}\}$. Because $d(A, B) > 4$ implies that $S_{A,2}$, $S_{B,2}$ are disjoint and because $X_{\mathbf{i}}$ are independent, the proposition follows. Similarly, a finite number D can be found such that the distance between any point and its kth nearest neighbor is smaller than D, yielding $R = 2D$.

We now turn our attention to the Delaunay triangulation and the Voronoi diagram, which have the same R. Arguing as before, the Voronoi polygon of a point $X \in \mathbf{i}$ is included in the sphere $S_{\mathbf{i},2}$ because the Voronoi polygon of X is the set of points that are nearer to the point X than any other point Y. Thus, if $X \in \mathbf{i}$ and $Y \in \mathbf{j}$ are Voronoi neighbors, then $d(\mathbf{i},\mathbf{j}) \leq 4$ because the midpoint of X and Y belongs to the Voronoi polygon of both. In fact, $d(\mathbf{i},\mathbf{j}) \leq 3$ because if $d(\mathbf{i},\mathbf{j}) \geq 4$, then any circle through X, Y has to contain a full subcube $\mathbf{r}$, where a point Z lies (because all subcubes are nonempty), and thus X, Y cannot be Voronoi neighbors.

The decision as to which of the points of the Poisson process belonging in subcubes in the sphere $S_{\mathbf{i},4}$ are Voronoi neighbors of $X \in \mathbf{i}$ is in no way affected by any point of the Poisson process outside $S_{\mathbf{i},4}$. Thus, arguing as in the nearest neighbor case, $R = 6$ is sufficient to prove the proposition. □

To apply Theorem 1, we need an upper bound on the length of $L_{\mathbf{i},n}$; this is provided in the following proposition.

PROPOSITION 4. *Conditionally on the event A_n there exists a constant h such that*

$$L_{\mathbf{i},n} \leq h \frac{\lambda^{1+1/2}}{n^{1/2}}. \tag{3}$$

PROOF. For each of the three problems studied, every distance is at most the length $\sqrt{2}/m$ of the diagonal of the subcube and there are at most $uN_{\mathbf{i},n}$

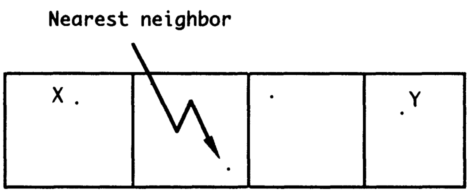

FIG. 3. $X_k \in \mathbf{i}$, $X_l \in \mathbf{j}$ *cannot be the nearest neighbors of each other if* $d(\mathbf{i},\mathbf{j}) \geq 3$.

edges, where the constant u depends on the problem. For the nearest neighbor there are $N_{\mathbf{i},n}$ edges, whereas for the Voronoi diagram and the Delaunay triangulation there are $3N_{\mathbf{i},n} - 6$ edges; see reference 13. Therefore, we have

$$L_{\mathbf{i},n} \le uN_{\mathbf{i},n}\frac{\sqrt{2}}{m}.$$

Given the event A_n, $N_{\mathbf{i},n} < \lceil e\lambda \rceil$, that is, $N_{\mathbf{i},n} \le \lceil e\lambda \rceil - 1 < e\lambda$, and because $m = (n/\lambda)^{1/2}$, we obtain (3), with $h = ue\sqrt{2}$. □

Finally, in order to apply Theorem 1 we need a lower bound on the variance of L_n. This is provided in the following proposition.

PROPOSITION 5. *There exists a constant* $f > 0$ *such that*

$$\mathrm{Var}[L_n] \ge f.$$

PROOF. We subdivide the cube $[0,1]^2$ into q^2 equal parts with $q = \lceil \sqrt{n} \rceil$. We identify a particular configuration that has a strictly positive probability and constant variability. We describe the construction for the nearest neighbor graph. Let $\varepsilon = 1/8\sqrt{n}$. Consider all the subcubes that have the following properties:

(i) They contain exactly two points inside a circle centered at the center of the subcube and radius ε, and no other points in the subcube.

(ii) For subcubes that do not touch the boundary, a ring of points exists outside but near the boundary of the subcube, each at most $\sqrt{5}\,\varepsilon$ from another such point. In particular, we demand that all the 36 smaller subcubes of dimension ε that surround the initial subcube be nonempty (see Figure 4). For subcubes that touch the boundary on one edge (not corner subcubes), we ask that all 26 smaller subcubes of dimension ε that surround the initial

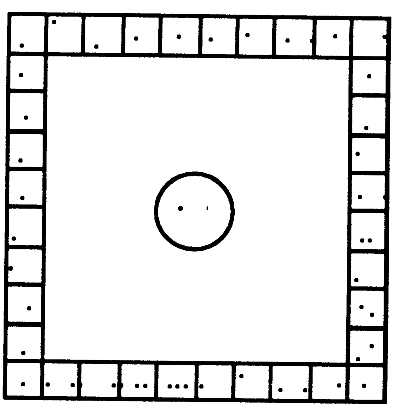

FIG. 4. *The configuration for the nearest neighbor graph.*

subcube be nonempty. Finally for subcubes that touch the boundary on two edges (corner subcubes) we ask that all 17 smaller subcubes of dimension ε that surround the initial subcube be nonempty.

Note that no point outside such a subcube has a nearest neighbor inside (because its distance to a point inside is at least 3ε and there are points outside within $\sqrt{5}\,\varepsilon$). Moreover, the two points inside the circle are nearest neighbors of each other. Their distance D has a certain variance $\sigma\varepsilon^2$ (σ is easily computable because it is the variance of the distance of two random points in a circle of radius 1). For subcubes that do not touch the boundary, the probability of such a configuration is $p = \exp(-n\pi\varepsilon^2)((n\pi\varepsilon^2)^2/2) \times \exp(-n(1/q^2 - \pi\varepsilon^2))(1 - e^{-n\varepsilon^2})^{36} \geq \exp(-n\pi\varepsilon^2)((n\pi\varepsilon^2)^2/2)\exp(-n(1/n - \pi\varepsilon^2))(1 - e^{-n\varepsilon^2})^{36} = 3.1779 \times 10^{-69} > 0$, because $n \leq q^2$ ($q = \lceil\sqrt{n}\,\rceil$) and $n\varepsilon^2 = 1/64$. Similarly, for subcubes that touch the boundary in one or two edges, the corresponding probabilities are p_1 and p_2. Obviously $p_2 > p_1 > p$.

Let K be the number of subcubes $I = \{i_1, \dots, i_K\}$ that satisfy the preceding two properties. Then, because there are four corner subcubes, $4(q-2)$ subcubes that touch the boundary in one edge and $(q^2 - 4(q-1))$ that do not touch the boundary, $E[K] = (q^2 - 4(q-1))p + 4(q-2)p_1 + 4p_2 > pq^2$, because $p_2 > p_1 > p$. Let F_n be the σ-algebra determined by the random set $\{i_1, \dots, i_K\}$ and by the positions of all the points lying outside the K subcubes. Then,

$$\operatorname{Var}[L_n] = \operatorname{Var}[E[L_n|F_n]] + E[\operatorname{Var}[L_n|F_n]] \geq E\left[\operatorname{Var}\left[\sum_{i\in I} d_i + \sum_{i\notin I} d_i|F_n\right]\right]$$

$$= E\left[\sum_{i\in I}\operatorname{Var}[d_i]\right] = \sigma\varepsilon^2 E[K] > \frac{\sigma p q^2}{64n} \geq \frac{\sigma p}{64} > 0,$$

where d_i is the sum of all the distances to their nearest neighbor of the points inside a subcube i. Letting $f = \sigma p/64$ proves the proposition for the nearest neighbor graph.

For the k-nearest graph the construction is similar except we require that there be $k+1$ points inside the circle, so that all k nearest neighbors are inside the circle and each of the outside subcubes contains $k+1$ points.

For the Voronoi diagram and the Delaunay triangulation we consider all subcubes with the following properties: They contain exactly one point inside a circle with center the center of the subcube and radius $\varepsilon/2$ and exactly three points inside a second circle of radius ε, and no other points in the subcube. The three points in the second circle are located as follows: We subdivide the circle into six equal sectors. The three points are located in alternate regions. As in the nearest neighbor graph, a ring of points exists outside the boundary at distance at most $\sqrt{5}\,\varepsilon$ from each other. The probability of such a configuration is $p_V > 0$ for subcubes not touching the boundary and larger than p_V for subcubes that touch the boundary. Note that the point inside has as Voronoi neighbors only the three points in the outside circle. Fixing the position of the three points in the outside circle, there is a certain variability σ_V/n, σ_D/n for the Voronoi polygon and its Delaunay triangulation of the point inside for some constants σ_V, σ_D. Letting T be the number of subcubes I satisfying the foregoing two properties, we let U_n be the σ-algebra determined by the random set $\{i_1, \dots, i_T\}$ and by the positions of all the points lying outside the T subcubes and inside the larger circle. Then,

$$\operatorname{Var}[L_n] = \operatorname{Var}[E[L_n|U_n]] + E[\operatorname{Var}[L_n|U_n]] \geq E\left[\operatorname{Var}\left[\sum_{i\in I} d_i + \sum_{i\notin I} d_i|U_n\right]\right]$$
$$= \frac{\sigma_V}{n}E[T] = \sigma_V p_V > 0,$$

where d_i is the sum of the Voronoi polygon (Delaunay triangulation) of the inside point. □

We now have all the ingredients to prove the CLT.

THEOREM 6. *The length L_n of the k nearest neighbor graph, the Voronoi diagram and the Delaunay triangulation satisfies the CLT*

$$\lim_{n\to\infty} P\left\{\frac{L_n - E[L_n]}{\sqrt{\operatorname{Var}[L_n]}} \leq x\right\} = \Phi(x), \tag{4}$$

where $\Phi(x)$ is the standard normal cumulative distribution function. Moreover, we have

$$\left|P\left\{\frac{L_n - E[L_n]}{\sqrt{\operatorname{Var}[L_n]}} \leq x\right\} - \Phi(x)\right| = O\left(\frac{(\log n)^{1+3/4}}{n^{1/4}}\right). \tag{5}$$

PROOF. We first establish that conditionally on the event A_n the random variable $(L_n - E[L_n])/\sqrt{\operatorname{Var}[L_n]}$ is asymptotically normal. Indeed, we define a dependency graph $G_n = (V_n, E_n)$ in the sense of Section 2.2. The set V_n consists of the subcubes $\mathbf{i}$ ($|V_n| = m^2$), whereas an edge $(\mathbf{i},\mathbf{j}) \in E_n$ if $d(\mathbf{i},\mathbf{j}) \leq R$, where R is defined in Proposition 3. The maximal degree D_n of this dependency graph satisfies $D_n \leq (2R+1)^2 - 1$, from Proposition 3. In addition, $B_n \leq h(\lambda^{1+1/2}/n^{1/2})$ from Proposition 4 and $\operatorname{Var}[L_n] \geq f$ from Proposition 5. Applying Theorem 1 and using $m^2 = n/\lambda$, we obtain

$$\frac{|V_n|D_n^2B_n^3}{\operatorname{Var}[L_n]^{3/2}} \leq \frac{m^2\left((2R+1)^2 - 1\right)^2\left[h\lambda^{1+1/2}/n^{1/2}\right]^3}{f^{3/2}} = z\frac{\lambda^{2+3/2}}{n^{1/2}},$$

where z is a constant. Because $\lambda < 2^2 c\log n$ for some $c > 5/4$, we obtain

$$\frac{|V_n|D_n^2B_n^3}{\operatorname{Var}[L_n]^{3/2}} \leq u\frac{\left[2^2c\log n\right]^{2+3/2}}{n^{1/2}} \to 0$$

as $n \to \infty$. Applying now Theorem 1, we have established that conditionally on the event A_n, $(L_n - E[L_n])/\sqrt{\operatorname{Var}[L_n]}$ is asymptotically normal N(0, 1).

We now need to show that the unconditional random variable $U_n = (L_n - E[L_n])/\sqrt{\mathrm{Var}[L_n]}$ is asymptotically normal. However,

$$P\{U_n \le x\} = P\{U_n \le x \mid A_n\}P\{A_n\} + P\{U_n \le x \mid A'_n\}P\{A'_n\}.$$

Taking limits and using Lemma 2 and the asymptotic normality of U_n given the event A_n we establish (4).

To establish the rate of convergence,

$$|P\{U_n \le x\} - \Phi(x)| \le P\{A_n\}|P\{U_n \le x \mid A_n\} - \Phi(x)| + P\{A'_n\}|P\{U_n \le x \mid A'_n\} - \Phi(x)|.$$

Applying (1) and (2) we obtain

$$|P\{U_n \le x\} - \Phi(x)| \le O\left(\frac{\lambda^{1+3/4}}{n^{1/4}}\right) + \left(1 - (1 - 2e^{-\lambda})^{n/\lambda}\right).$$

Because $\lambda \ge c \log n$ and $c > 5/4$, we have

$$\begin{aligned} |P\{U_n \le x\} - \Phi(x)| &\le O\left(\frac{(\log n)^{1+3/4}}{n^{1/4}}\right) + \left(1 - \exp\left(-\frac{2}{cn^{c-1}\log n}\right)\right) \\ &= O\left(\frac{(\log n)^{1+3/4}}{n^{1/4}}\right). \end{aligned}$$ □

REMARK 1. One might ask whether the rates of convergence (5) that our method gives are the best possible. In our derivation of the rates we chose $\lambda = \Theta(\log n)$. This is the best choice of λ because in order for $\lim_{n\to\infty} P\{A_n\} = 1$ in Lemma 2, we need $\lambda \ge c \log n$. If λ grows more slowly than $\log n$, then $\lim_{n\to\infty} P\{A_n\} = 0$ and our method fails. We suspect, however, that the logarithmic terms that appear in (5) are not essential but rather a by-product of the method; that is, we conjecture

$$|P\{U_n \le x\} - \Phi(x)| \le O\left(\frac{1}{n^{1/4}}\right).$$

REMARK 2. One advantage of the Baldi and Rinott [4] dependency graph theorem is that it yields rates of convergence. For another application of the theorem, we consider the two CLTs for the number of points in the convex hull and for the area outside the convex hull of points generated according to a Poisson process of intensity n in a circle of radius 1, obtained by Groeneboom [8] and Hsing [9], respectively. Their approach is similar to ours and is equivalent to dividing the circle into equal sectors S_i of angle θ_n, where θ_n is chosen such that each slice R_i, $i = 1, \dots, 2\pi/\theta_n$, bounded on one side by the circle and on the other side by the chord connecting two consecutive partition points, contains an expected number of $c \log n$ points. The area of R_i (see Figure 5) can be calculated as the difference of the area of the sector S_i and the area of the triangle AOB, which is $(\theta_n - \sin\theta_n)/2 \le \theta_n^3/12$ because $\sin\theta_n \ge \theta_n - \theta_n^3/3!$. Therefore, because the area of R_i is $O(\theta_n^3)$, we obtain

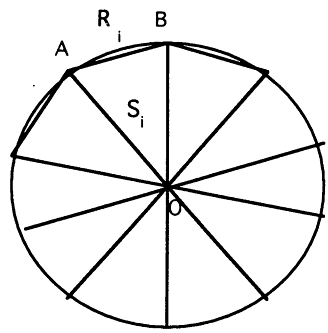

FIG. 5. *The partition for the convex hull problem.*

that $\theta_n = O((\log n/n)^{1/3})$. Then c is chosen so that the probability that all these slices contain at least one point and at most $2c \log n$ points converges to 1 as $n \to \infty$.

The key observation is that conditioned on all the slices being nonempty, both the number of vertices in the part of the convex hull lying in a sector S_i and the area outside the convex hull, inside a sector S_i, depend only on the configuration of points inside the sectors S_{i-1}, S_i, S_{i+1}. This implies that the dependency graph between the parts X_i of the convex hull lying in the sector S_i has degree $D_n = 4$. Using the formula for the rate from Theorem 1, $r_n = (|V_n| D_n^2 B_n^3 / \sigma_n^3)^{1/2}$, with $|V_n| = 2\pi/\theta_n$, $\sigma_n = \Theta(n^{1/6})$ (Theorem 3.4 of Groeneboom [8]), $B_n = O(\log n)$ for the number of points in the convex hull and $|V_n| = 2\pi/\theta_n$, $\sigma_n = \Theta(n^{-5/6})$ (Theorem 3.1 of Hsing [9]), $B_n = O(\theta_n^3)$ for the area outside the convex hull, we obtain the rate of convergence of

$$r_n = O\big((\log n)^{4/3}/n^{1/12}\big)$$

for both the number of points in the convex hull and the area outside the convex hull.

REMARK 3. Another advantage of Theorem 1 is that it only requires bounding the variance below and thus can be used in problems where the exact computation of the variance is not available.

3. Expected lengths. In this section we briefly review the known results for the expected lengths of the graphs we considered.

THEOREM 7 (Miles [11]). *The expected length of the k nearest neighbor graph $N_{k,n}$ satisfies*

$$\lim_{n\to\infty} \frac{E[L(N_{k,n})]}{n^{1/2}} = \frac{1}{2\pi^{1/2}} \sum_{j=1}^{k} \frac{\Gamma(j + \frac{1}{2} - 1)}{(j-1)!}.$$

The expected length of the Voronoi diagram Vor_n *and the Delaunay triangulation* Del_n *satisfies*

$$\lim_{n\to\infty} \frac{E[L(\mathrm{Vor}_n)]}{n^{1/2}} = 2,$$

$$\lim_{n\to\infty} \frac{E[L(\mathrm{Del}_n)]}{n^{1/2}} = \frac{32}{3\pi}.$$

For direct geometric proofs of these results the reader is referred to Avram and Bertsimas [1].

4. Concluding remarks. Our approach generalizes easily in dimension d for the lengths of the k nearest neighbor graph and the Delaunay triangulation. The rate of convergence in this case becomes

$$\left|P\left\{\frac{L_n - E[L_n]}{\sqrt{\mathrm{Var}[L_n]}} \le x\right\} - \Phi(x)\right| = O\left(\frac{(\log n)^{1+3/2d}}{n^{1/4}}\right).$$

The CLTs were established for the Poisson model. It would be desirable to establish the CLTs for the usual Euclidean model, in which n points X_i, $i = 1,\dots,n$, are uniformly and independently distributed in a d-dimensional cube $[0,1]^d$.

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